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MULTI-CRITERIA DECISION MAKING UNDER HESITANT FUZZY ENVIRONMENT

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ABSTRACT

Dealing with uncertainty has always been a challenging problem and different tools have been proposed to deal with it. Recently, a new model that is based on hesitant fuzzy sets has been presented to manage situations in which experts hesitate between several values to assess an alternative. Hesitant fuzzy set is a very useful technique in situations where there are some difficulties in determining the membership of an element to a set. Most aggregation methods are under the assumption that the criteria are independent. However, in real decision making problems, there exists correlation between criteria. The Heronian mean (HM) can capture the interrelationship between input arguments. Combining the HMs and hesitant fuzzy sets, some new hesitant fuzzy Heronian means (HFHMs) are explored. In this paper, multi-criteria decision making problems where the criteria are correlative under hesitant fuzzy environment are investigated. First, we propose the Hesitant Fuzzy Heronian Mean (HFHM). Some of its desirable properties and special cases ofHFHM are studied in detail. Furthermore, the HFHM is extended to the generalized Hesitant Fuzzy Heronian Mean (GHFHM). The generalized weighted Hesitant Fuzzy Heronian Mean (GWHFHM) is developed to give due weightage to the arguments. Finally, a real life situation involving the portfolio management planning is provided to illustrate the usefulness of developed HFHM operator.

KEYWORDS: Hesitant Fuzzy Set, Heronian Mean, Hesitant Fuzzy Heronian Mean

1. INTRODUCTION

The management of uncertain or vague information in real world problems is always a challenging task. The Hesitant Fuzzy Set (HFS) can be used to efficiently manage the situation where people hesitate between several possible values to express their opinions. HFS is a novel and recent extension of fuzzy sets (FS) to model the uncertainty originated by the hesitation that might arise in the assignment of membership degrees of the elements to a fuzzy set. Hence it is a more powerful tool to deal with hesitancy and uncertainty in real applications than Zadeh's fuzzy set. The FS theory has been extended to many other forms such as interval-valued fuzzy set (Zadeh, 1975), type-2 fuzzy set (Dubois and Prade, 1980; Miyamoto, 2005), fuzzy multiset (Yager, 1986; Miyamoto, 2000) and intutionistic fuzzy set (Atanassov, 1986). Torra and Narukawa (2009) and Torra (2010) first proposed the concept of hesitant fuzzy set (HFS). The concept of HFS facilitates in dealing with uncertainty caused by hesitation.

The main problem in multi-criteria decision making (MCDM) problems is the aggregation of the individual criteria to obtain a measure of satisfaction to the overall collection of criteria. In this regard the aggregation techniques play an important role.

The aggregation operators (Beliakov et al, 2007; Torra and Narukawa, 2007; Grabisch et al, 2009) are common aggregation techniques to fuse all the input individual arguments into a single one.

Zhu et al. (2012) investigated the geometric Bonferroni mean combining the Bonferroni mean and the geometric mean under hesitant fuzzy environment. Xia and Xu (2011) presented some hesitant fuzzy operational laws based on the relationship between the HFEs and the IFVs. In order to aggregate hesitant fuzzy information, Xia and Xu proposed some Algebraic t- conorm and t-norm based operational laws for HFSs, based on which, a variety of hesitant fuzzy aggregation operators have been developed in recent years. By extending the Bonferroni mean (BM) to hesitant fuzzy environments, Zhu and Xu (2013) developed the hesitant fuzzy Bonferroni means (HFBMs) and the weighted hesitant fuzzy Bonferroni mean (WHFBM). In order to consider the relationship between the hesitant fuzzy input arguments, Zhang (2013) developed several new hesitant fuzzy aggregation operators, including the hesitant fuzzy power average (HFPA) operator, the hesitant fuzzy power geometric (HFPG) operator, the generalized hesitant fuzzy power average (GHFPA) operator, the generalized hesitant fuzzy power geometric (GHFPG) operator.

Heronian Mean (HM) is a mean type aggregation technique which is developed to reflect the interrelationship of individual criteria. The HM operator focuses on the aggregated arguments, whereas the choquet integral or the power average focuses on changing the weight vector of the aggregation operators. HM aggregate only the crisp numbers restricting its applications to more extensive fields. As an effective way to solve this problem, the HM is extended to the generalized hesitant fuzzy environment which is the focus of this paper. In Section 2, we briefly review some basic concepts. Section 3 proposes the Hesitant Fuzzy Heronian Meanto aggregate the hesitant fuzzy information, whose desirable properties and some special cases are also studied. In Section 4 Generalized Hesitant Fuzzy Heronian Mean is introduced. In Section 5, we develop a method for multi-criteria group decision making based on the HFHM under hesitant fuzzy environment. Section 6 gives a numerical example to validate the effectiveness of the proposed HFHM.

2. PRELIMINARIES (TORRA V, NARUKAWA Y)

Definition

Let X be a fixed set, an Hesitant Fuzzy Set (HFS) on X is in terms of a function that when applied to X returns a subset of [0, 1]. Mathematically Xia and Xu express the HFS as, $H = \left\{ \left\langle x, h_H(x) \right\rangle / x \in X \right\}$ where $h_H(x)$ is a set of some values in [0, 1] denoting the possible membership degrees of the element $x \in X$ to the set H. $h_H(x)$ is called the hesitant fuzzy element (HFE)

2.1 Operations on HFEs Xia and Xu

Let h, h_1 *and* h_2 be three HFEs. Then

(1) $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$

(2) $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\}$

(3) $h^\lambda = \bigcup_{\gamma \in h} \{\gamma^\lambda\}$

(4) $\lambda h = \bigcup_{\gamma \in h} \{1-(1-\gamma_1)^{\lambda}\}, \quad \lambda > 0$

Xia and Xu defined the score function to compare two HFEs as below.

2.2 Definition: Score Function

Let h be an HFE. Then $s(h) = \frac{1}{n(h)} \sum_{\gamma \in h} \gamma$ is called the score function of h, where n(h) is the number of values of h. If $s(h_1) > s(h_2)$ then $h_1 > h_2$ and if $s(h_1) = s(h_2)$ then $h_1 = h_2$

2.3 Definition: Accuracy Function

Let h be an HFE; then $k(h) = \sqrt{\frac{1}{n(h)} \sum_{\gamma \in h} (\gamma - s(h))^2}$ is called the accuracy function of h wheren(h) is the number of values in h and *s(h)* is the score function of h.

2.4 Definition: (Beliakov 2007)

Let p, q ≥ 0, and $a_i \ (i = 1, 2, ..., n)$ be a collection of non-negative numbers.

If $H^{p,q}(a_1, a_2, ..., a_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}}$ then $H^{p,q}$ is called the Heronian Mean (HM)

HM has the following properties

(1) $H^{p,q}(0, 0, ..., 0) = 0$

(2) $H^{p,q}(a \ , a \ , ..., a) = a$ if $a_i = a$ for all i

(3) $H^{p,q}(a_1, a_2, ..., a_n) \ \geq \ H^{p,q}(b_1, \mathrm{b}_2, ..., \mathrm{b}_n)$ if $a_i \geq b_i$ for all i.That is, H is monotonic.

(4) $\min\{a_i\} \ \leq \ H^{p,q}(a_1, a_2, ..., a_n) \ \leq \ \max\{a_i\}$

By combining the definition of Heronian Mean and HFE, the hesitant fuzzy Heronian mean is developed as follows:

2.5 Definition

Let $h_i \ (i = 1, 2, ..., n)$ be a collection of HFEs. For any $p, q > 0$, if

$$HFHM^{p,q}(h_1, \mathrm{h}_2, ..., \mathrm{h}_n) = \left(\frac{2}{n(n+1)} \left(\bigoplus_{i,j=1}^{n} \left((h_i)^p \otimes (h_j)^q \right) \right) \right)^{\frac{1}{p+q}}$$

Then $HFHM^{p,q}$ is called as the hesitant fuzzy Heronian Mean. Based on the above definition, we derive the following theorem.

2.6 Theorem

Let p, q > 0 and let $h_i\,(i=1,2,...,n)$ be a collection of HFEs.Then the aggregated value by using the HFHM

Is also an HFE and $HFHM^{p,q}(h_1,h_2,...,h_n) = \bigcup_{\gamma_{ij}\in f_{ij}} \left\{ \left(1-\prod_{i,j=1}^{n} \left(1-\gamma_{i,j}\right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\} \rightarrow (1)$

Where $f_{ij} = h_i^p \otimes h_j^q$ reflects the inter-relationship between the HFEs h_i and h_j , $i, j = 1,2,...,n$

Proof

$$f_{ij} = h_i^p \otimes h_j^q = \bigcup_{\gamma_{ij}\in f_{ij}} \{\gamma_{ij}\} = \bigcup_{\gamma_i\in h_i,\gamma_j\in h_j} \{\gamma_i^p \gamma_j^q\}$$ is an HFE.

Hence from equation (1) $HFHM^{p,q}(\mathrm{h}_1,\mathrm{h}_2,...,\mathrm{h}_n) = \left(\frac{2}{n(n+1)} \left(\bigoplus_{i,j=1}^{n} f_{ij} \right) \right)^{\frac{1}{p+q}}$

By operational laws of HFEs $\frac{2}{n(n+1)} \left(\bigoplus_{i,j=1}^{n} f_{ij} \right) = \bigoplus_{i,j=1}^{n} \left(\frac{2}{n(n+1)} f_{ij} \right)$

$$= \bigcup_{\gamma_{ij}\in f_{ij}} \left\{ \left(1-\prod_{i,j=1}^{n} \left(1-\gamma_{i,j}\right)^{\frac{2}{n(n+1)}} \right) \right\}$$

Therefore, $HFHM^{p,q}(\mathrm{h}_1,\mathrm{h}_2,...,\mathrm{h}_n) = \left(\frac{2}{n(n+1)} \left(\bigoplus_{i,j=1}^{n} f_{ij} \right) \right)^{\frac{1}{p+q}}$

$= \bigcup_{\gamma_{ij}\in f_{ij}} \left(1-\prod_{i,j}^{n} \left(1-\gamma_{ij}\right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}}$ where $f_{ij} = h_i^p \otimes h_j^q = \bigcup_{\gamma_{i}\in h_i,\gamma_j\in h_j} \left(\gamma_i^p \gamma_j^q\right)$ Hence the proof.

Some properties of $HFHM^{p,q}$ is discussed here.

2.7 Theorem

Let $h_a = \left(h_{a_1}, h_{a_2}, ..., h_{a_n}\right)$ and $h_b = \left(h_{b_1}, h_{b_2}, ..., h_{b_n}\right)$ be two collections of HFEs, let $\gamma_{a_i} \in h_{a_i}, \gamma_{b_i} \in h_{b_i}$ such that $\gamma_{a_i} \le \gamma_{b_i}$ for all i and $f_{ij} = h_i^p \otimes h_j^q = \bigcup_{\gamma_{ij}\in f_{ij}} \{\gamma_{ij}\} = \bigcup_{\gamma_i\in h_i,\gamma_j\in h_j} \{\gamma_i^p \gamma_j^q\}$

Proof

Since $\gamma_{a_i} \le \gamma_{b_i}$ for any $\gamma_{a_i} \in h_{a_i}$, $\gamma_{b_i \in h_{b_i}}$, $\gamma_{a_i}\gamma_{a_j} \le \gamma_{b_i}\gamma_{b_j}$

Also for any $\eta_{a_{ij}} \in f_{aij}$, $\eta_{b_{ij}} \in f_{bij}$, $\eta_{a_{ij}} = \gamma_{a_i}^p \gamma_{a_j}^q \le \eta_{b_{ij}} = \gamma_{b_i}^p \gamma_{b_j}^q$

Hence $\left(1-\prod_{i,j=1}^{n}\left(1-\eta_{f_{ij}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \le \left(1-\prod_{i,j=1}^{n}\left(1-\eta_{b_{ij}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}$

According to definition and theorem 2.7, $HFHM^{p,q}\left(h_{a_1}, a_{a_2}, \ldots, a_{a_n}\right) = \left(\frac{2}{n(n+1)}\left(\bigoplus_{i,j=1}^{n} f_{a_{ij}}\right)\right)^{\frac{1}{p+q}}$

$$= \bigcup_{\eta_{a_{ij}} \in f_{ij}} \left(1-\prod_{i,j=1}^{n}\left(1-\eta_{a_{ij}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \le \bigcup_{\eta_{b_{ij}}} \in f_{ij} \left(1-\prod_{i,j=1}^{n}\left(1-\eta_{b_{ij}}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}$$

$$= \left(\frac{2}{n(n+1)}\left(\bigoplus_{i,j=1}^{n} f_{b_{ij}}\right)\right)^{\frac{1}{p+q}} = HFHM^{p,q}\left(h_{b_1}, h_{b_2}, \ldots, h_{b_n}\right)$$

2.8 Theorem

Let $h_i\left(i=1,2,\ldots,n\right)$ be a collection of HFEs, $h_i^+ = \bigcup_{\gamma_i \in h_i} \max\{\gamma_i\}$

$h_i^- = \bigcup_{\gamma_i \in h_i} \min\{\gamma_i\}$, $\gamma_i^+ \in h_i^+$, $\gamma_i^- \in h_i^-$ and $f_{ij} \in h_i^p \otimes h_j^q = \bigcup_{\eta_{ij} \in f_{ij}} \{\eta_{ij}\}$

$= \bigcup_{\gamma_i \in h_i, \gamma_j \in h_j} \{\gamma_i^p \gamma_j^q\}$, then $h_i^- \le HFHM^{p,q}\left(h_1, h_2, \ldots, h_n\right) \le h_i^+$

Proof

Since $\gamma_i^- \le \gamma_i \le \gamma_i^+$ for all i, $\left(\gamma_i^-\right)^{p+q} \le \gamma_i^p \gamma_j^q \le \left(\gamma_i^+\right)^{p+q}$

$\left(\prod_{i,j=1}^{n}\left(1-\eta_{ij}\right)\right)^{\frac{2}{n(n+1)}} \ge \left(\prod_{i,j=1}^{n}\left(1-\left(\gamma_i^+\right)^{p+q}\right)\right)^{\frac{2}{n(n+1)}}$ and $\left(\prod_{i,j=1}^{n}\left(1-\eta_{ij}\right)\right)^{\frac{2}{n(n+1)}} \le \left(\prod_{i,j=1}^{n}\left(1-\left(\gamma_i^-\right)^{p+q}\right)\right)^{\frac{2}{n(n+1)}}$

Hence, $\left(1-\left(\prod_{i,j=1}^{n}\left(1-\eta_{ij}\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \ge \left(1-\left(1-\left(\gamma_i^-\right)^{p+q}\right)\right)^{\frac{1}{p+q}} = \gamma_i^-$ and

$$\left(1-\left(\prod_{i,j=1}^{n}\left(1-\eta_{ij}\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \leq \left(1-\left(1-\left(\gamma_i^{+}\right)^{p+q}\right)\right)^{\frac{1}{p+q}} = \gamma_i^{+}$$

By definition,

$$\bigcup_{\eta_{11}\in f_{11},\eta_{12}\in f_{12},\eta_{21}\in f_{21},\ldots,\eta_{n(n-1)}\in f_{n(n-1)}} \left\{\left(1-\prod_{i,j=1}^{n}\left(1-\eta_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right\} \geq \bigcup_{\eta_{ij}^{-}\in f_{ij}^{-}} \left\{\left(1-\left(1-\eta_{ij}^{-}\right)\right)^{\frac{1}{p+q}}\right\}$$

$$= \bigcup_{\eta_{ij}^{-}\in f_{ij}^{-}} \left\{\left(\eta_{ij}^{-}\right)^{\frac{1}{p+q}}\right\} \quad = \bigcup_{\gamma_i^{-}\in h_i^{-}} \left\{\gamma_i^{-}\right\} = f_i^{-}$$

Similarly,

$$\bigcup_{\eta_{11}\in f_{11},\eta_{12}\in f_{12},\eta_{21}\in f_{21},\ldots,\eta_{n(n-1)}\in f_{n(n-1)}} \left\{\left(1-\prod_{i,j=1}^{n}\left(1-\eta_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right\} \leq \bigcup_{\eta_{ij}^{+}\in f_{ij}^{+}} \left\{\left(1-\left(1-\eta_{ij}^{+}\right)\right)^{\frac{1}{p+q}}\right\}$$

$$= \bigcup_{\eta_{ij}^{+}\in f_{ij}^{+}} \left\{\left(\eta_{ij}^{+}\right)^{\frac{1}{p+q}}\right\} = \bigcup_{\gamma_i^{+}\in h_i^{+}} \left\{\gamma_i^{+}\right\} = f_i^{+}$$

Hence the proof

2.9 Some Special Cases of HFHMs

Some special cases of HFHMs by changing the parameters are discussed below.

Case 1: If $q \to 0,$ we obtain a generalized hesitant fuzzy mean (GHFM):

$$HFHM^{p,0}\left(h_1, \mathrm{h}_2, \ldots, \mathrm{h}_n\right) = \left(\frac{1}{n}\left(\bigoplus_{i,j=1}^{n}(h_i)^p\right)\right)^{\frac{1}{p}} = \bigcup_{\gamma_i\in h_i}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right\}$$

Case 2: If $p = 2$ and $q = 0$ we obtain a hesitant fuzzy square mean:

$$HFHM^{2,0}\left(h_1, \mathrm{h}_2, \ldots, \mathrm{h}_n\right) = \left(\frac{1}{n}\left(\bigoplus_{i,j=1}^{n}(h_i)^2\right)\right)^{\frac{1}{2}} = \bigcup_{\gamma_i\in h_i}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_i^{2}\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right\}$$

Case 3: If $p = 1$ and $q \to 0,$ we obtain a hesitant fuzzy averaging (HFA) operator (Xia and Xu, 2011):

$$HFHM^{1,0}\left(h_1, \mathrm{h}_2, \ldots, \mathrm{h}_n\right) = \frac{1}{n}\left(\bigoplus_{i,j=1}^{n}(h_i)\right) = \bigcup_{\gamma_i\in h_i}\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\frac{1}{n}}\right)\right\}$$

Case 4: If $p = q = 1$ we obtain a hesitant fuzzy inter-related square mean (HFISM):

$$HFHM^{1,1}\left(h_1, \mathrm{h}_2, \ldots, \mathrm{h}_n\right) = \left(\frac{2}{n(n+1)}\left(\underset{i,j=1}{\overset{n}{\oplus}}\left((h_i) \otimes (h_j)\right)\right)\right)^{\frac{1}{2}} = \bigcup_{\gamma_{ij} \in f_{ij}} \left\{\left(1 - \prod_{i,j=1}^{n}\left(1 - \gamma_{i,j}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{2}}\right\}$$

Where $f_{ij} = h_i^p \otimes h_j^q \ (i, j = 1, 2, \ldots, n)$

3. THE GENERALIZED HESITANT FUZZY HERONIAN MEAN

The Heronian mean proposed by Beliakov is further extended to hesitant fuzzy environment.

3.1 Definition: Let p, q, r≥ 0 and $a_1 \left(i = 1, 2, \ldots, n\right)$ be a collection of non-negative numbers.

If, $GHM^{p,q,r}\left(a_1, a_2, \ldots, a_n\right) = \left(\frac{2}{n(n+1)(n+2)} \sum_{i,j,k=1}^{n} a_i^p a_j^q a_k^r\right)^{\frac{1}{(p+q+r)}}$ then $GHM^{p,q,r}$ is called the Generalized Heronian Mean Operator.

Based on the above definition the generalized hesitant fuzzy Heronian Mean Operator is proposed.

3.2 Definition

Let $h_i = \left(i = 1, 2, \ldots, n\right)$ be a collection of HFEs on X. for any $p, q, r > 0$, is

$GHFHM^{p,q,r}\left(h_1, \mathrm{h}_2, \ldots, \mathrm{h}_n\right) = \frac{2}{n(n+1)(\mathrm{n}+2)}\left(\underset{i,j,k=1}{\overset{n}{\oplus}} (h_i) \otimes \left(h_j\right)^q \otimes \left(h_k\right)^r\right)^{\frac{1}{p+q+r}}$ then $GHM^{p,q,r}$ is called the generalized hesitant fuzzy Heronian Mean (GHFHM) operator. Based on the operations of HFEs we derive the following theorem.

3.3 Theorem

Let p, q, r > 0 and let $\left(i = 1, 2, \ldots, n\right)$ be a collection of HFEs on X, then aggregated value by using the GHFHM is also an HFE and $GHM^{p,q,r}\left(h_1, h_2, \ldots, h_n\right) = \bigcup_{\gamma_{i,j,k}} \left(1 - \prod_{i,j,k=1}^{n}\left(1 - \gamma_{i,j,k}\right)^{\frac{2}{n(n+1)(n+2)}}\right)^{\frac{1}{p+q+r}}$ where

$f_{i,k,k} = h_i^p \otimes h_j^q \otimes h_k^r$

Proof

Based on the operations of HFEs we have $h_i^p = \bigcup_{\gamma_i \in h_i} \left\{\gamma_i^{\,p}\right\}, \mathrm{h}_j^q = \bigcup_{\gamma_j \in h_j} \left\{\gamma_j^{\,q}\right\}, h_k^q \bigcup_{\gamma_k \in h_k} \left\{\gamma_k^r\right\}$ and

$h_i^p h_j^q h_k^r = \bigcup_{\gamma_i \in h_i, \gamma_j \in h_j, \gamma_k \in h_k} \left\{\gamma_i^p \gamma_j^q \gamma_k^r\right\}$

$$\therefore\ f_{i,j,k} = (h_i)^p \otimes (h_j)^q \otimes (h_k)^r = \bigcup_{\gamma_j \in h_j} \{\gamma_j{}^q\}, h_k^q \bigcup_{\gamma_k \in h_k} \{\gamma_k^r\}$$

$$= \bigcup_{\gamma_{i,j,k} \in f_{i,j,k}} \{\gamma_{i,j,k}\}$$

By definition we have $GHFHM^{p,q,r}(h_1, \mathrm{h}_2, ..., \mathrm{h}_n) = \left(\frac{2}{n(n+1)(\mathrm{n}+2)} \left(\bigoplus_{i,j,k=1}^{n} (h_i) \otimes (h_j)^q \otimes (h_k)^r \right)\right)^{\frac{1}{p+q+r}}$

$$= \left(\frac{2}{n(n+1)(n+2)} \left(\bigoplus_{i,j,k=1}^{n} f_{i,j,k} \right) \right)^{\frac{1}{p+q+r}}$$

Based on the operations of HFEs $\frac{2}{n(n+1)(n+2)} \left(\bigoplus_{i,j,k=1}^{n} f_{i,j,k} \right) = \bigcup_{\gamma_{i,j,k} \in f_{i,j,k}} \left\{ 1 - (1-\gamma_{i,j,k})^{\frac{2}{n(n+1)(n+2)}} \right\}$

Hence $\frac{2}{n(n+1)(n+2)} \left(\bigoplus_{i,j,k=1}^{n} f_{i,j,k} \right) = \bigcup_{\gamma_{i,j,k} \in f_{i,j,k}} \left\{ \left(1 - \prod_{i,j,k=1}^{n} (1-\gamma_{i,j,k})^{\frac{2}{n(n+1)(n+2)}} \right)^{\frac{1}{p+q+r}} \right\}$

Also for all i, j k, $0 \le \left(1 - \prod_{i,j,k=1}^{n} (1-\gamma_{i,j,k})^{\frac{2}{n(n+1)(n+2)}} \right)^{\frac{1}{p+q+r}} \le 1$

Hence the proof of the Theorem

The properties of GHFHM operator is stated below:

Proposition 1: I dempotency

Let $h_i\ (i = 1, 2, ..., n)$ and $f_i\ (i = 1, 2, ..., n)$ be two collections of HFEs on X. If for i, $\gamma_i = \gamma,$ where γ_i the elements of hesitant are fuzzy set h then $GHFHM^{p,q,r}(\mathrm{h}_1, \mathrm{h}_2, ..., \mathrm{h}_i) = h$

Proposition 2: Monotonicity

Let $h_i\ (i = 1, 2, ..., n)$ and $f_i\ (i = 1, 2, ..., n)$ be a collection of HFEs on X. Let ε_i be the elements of hesitant fuzzy set h_i and γ_i be the elements of the hesitant fuzzy set f_i. If $\varepsilon_i \le \gamma_i$ for all i then $GHFHM^{p,q,r}(\mathrm{h}_1, \mathrm{h}_2, ..., \mathrm{h}_n) \le GHFHM^{p,q,r}(f_1, f_2, ..., f_n)$

Proposition 3: Boundedness

Let $h_i\ (i = 1, 2, ..., n)$ be a collection of HFEs on X

Let $h^+ = \bigcup_{\gamma_i \in h_i} \max_{i=1}^{n} \{\varepsilon_i\}$ and $h^+ = \bigcup_{\gamma_i \in h_i} \min_{i=1}^{n} \{\varepsilon_i\}$

Then $h^- \le GHFHM^{p,q,r}(h_1, h_2, ..., h_n) \le h^+$

Proposition 4: Commutativity

Let $h_i\,(i = 1, 2, ..., n)$ be a collection of HFEs on X. Let $(h_1^{'}, h_1^{'}, ..., h_n^{'})$ be any permutation of $(h_1, \mathrm{h}_2, ..., \mathrm{h}_n)$.
Then $GHFHM^{p,q,r}(\mathrm{h}_1^{'}, \mathrm{h}_2^{'}, ..., \mathrm{h}_i^{'}) = GHFHM^{p,q,r}(\mathrm{h}_1, \mathrm{h}_2, ..., \mathrm{h}_n)$

3.4 Some Special Cases of GHFHM Operators

Case (1) If $p \to 0$ then GHFHM reduces to $GHFHM^{p,q,r}(\mathrm{h}_1, \mathrm{h}_2, ..., \mathrm{h}_n)\ \lim_{p \to 0} GHFHM^{p,q,r}(\mathrm{h}_1, \mathrm{h}_2, ..., \mathrm{h}_n) = HFHM^{q,r}(\mathrm{h}_1, \mathrm{h}_2, ..., \mathrm{h}_n)$

$$= \bigcup_{\gamma_j \in h_j, \gamma_k \in h_k} \left(\frac{2}{n(n+1)} \left(\bigoplus_{j,k=1}^{n} h_j^q \otimes h_k^r \right) \right)^{\frac{1}{q+r}} = \bigcup_{\delta_{j,k} \in g_{j,k}} \left\{ \left(\left(1 - \delta_{j,k}\right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{q+r}} \right\}$$ which is called Hesitant fuzzy Heronian mean (HFHM) where $g_{j,k} = h_j^q \otimes h_k^r$

Case (2) If $p = 1,\ \ q \to 0\ \ and\ \ r \to 0$ then the GHFHM reduces to the hesitant fuzzy average

$$GHFHM^{1,0,0}(\mathrm{h}_1, \mathrm{h}_2, ..., \mathrm{h}_n) = \frac{1}{n}\left(\bigoplus_{i=1}^{n} h_i \right) = \bigcup_{\gamma_i \in h_i} \left\{ \left(1 - \prod_{i=1}^{n} (1 - \gamma_i)^{\frac{1}{n}} \right) \right\}$$

Case (3) If $q \to 0$ and $r \to 0$ then GHFHM reduces to the hesitant fuzzy geometric average.

$$\lim_{q \to 0, r \to 0} GHFHM^{p,q,r}(\mathrm{h}_1, \mathrm{h}_2, ..., \mathrm{h}_n) = \left(\frac{1}{n} \left(\oplus h_i^p \right) \right)^{\frac{1}{p}}$$

$$= \bigcup_{\gamma_i \in h_i} \left\{ \left(1 - \prod_{i=1}^{n} \left(1 - \varepsilon_i^p\right)^{\frac{1}{n}} \right)^{\frac{1}{p}} \right\}$$

Case (4) if $p = q = r = 1$, then the GHFHM reduces to $GHFHM^{1,1,1}(\mathrm{h}_1, \mathrm{h}_2, ..., \mathrm{h}_n)$

$$= \bigcup_{\delta_{i,j,k} \in d_{i,j,k}} \left\{ \left(1 - \prod_{i,j,k=1}^{n} \left(1 - \delta_{i,j,k}\right)^{\frac{2}{n(n+1)(n+2)}} \right)^{\frac{1}{3}} \right\}$$ which is called the Generalized Hesitant Fuzzy Inter-related Square Mean (GHFISM) and $d_{i,j,k} = h_i \otimes h_j \otimes h_k$

3.5 The Generalized Weighted Hesitant Fuzzy Heronian Mean GWHFHM

To consider the importance of aggregated arguments a GWHFM is developed as follows.

Definition

Let $p,q,r>0$, $h_i\left(i=1,2,...n\right)$ be a collection of HFEs, $\omega=(\omega_1,\omega_2,...\omega_n)^T$ be the weight vector of $h_i^{'}s$, ω_i indicates the importance degree of h_i, satisfying $\omega_i\in\left[0,1\right]\left(i=1,2,...,n\right)$ and $\sum_{i=1}^{n}\omega_i=1$ If

$$GHFHM_{\omega}^{p,q,r}\left(\mathrm{h}_1,\mathrm{h}_2,...,\mathrm{h}_n\right)=\left(\frac{2}{n(n+1)(n+2)}\underset{i,j,k=1}{\overset{n}{\oplus}}\left((\omega_i h_i)^p\otimes(\omega_j h_j)^q\otimes(\omega_k h_k)^r\right)\right)^{\frac{1}{p+q+r}}$$

Then $GWHFH_{\omega}$ is called the generalized weighted hesitant fuzzy Heronian Mean Operator. Based on the above definition we develop the following theorem.

Theorem

Let $p,q,r>0$ and $h_i\left(i-1,2,...,n\right)$ be a collection of HFEs, whose weight vector is $\omega=\left(\omega_1,\omega_2,...,\omega_n\right)^T$ which satisfies $\omega_i\in\left[0,1\right]\left(i=1,2,...,n\right)$ and $\sum_{i=1}^{n}\omega_i=1$ and the aggregated value by using the GWHFH is a HFE; and

$GWHFH^{p,q,r}\left(h_1,h_2,...,h_n\right)$

$$=\bigcup_{\eta_{i,j,k}^{\omega}\in\sigma_{i,j,k}^{\omega}}\left\{\left(1-\prod_{i,j,k=1}^{n}\left(1-\eta_{ij}^{\omega}\right)^{\frac{2}{n(n+1)(n+2)}}\right)^{\frac{1}{p+q+r}}\right\}$$

where $\sigma_{i,j,k}^{\omega}=\left(\omega_i h_i\right)^p\otimes\left(\omega_j h_j\right)^q\otimes\left(\omega_k h_k\right)^r$ reflects the inter-relationship between h_i,h_j *and* $h_k\left(i,j,k=1,2,...,n\right)$.

GWHFH operators satisfies the following properties: (1) Monotonicity (2) Idempotency (3) Boundedness and (4) Commutativity

4. MCDM METHOD BASED ON HFHM

Step 1: Identify the set of available alternatives and criteria.

Let $y=\left\{y_1,y_2,...,\mathrm{y}_n\right\}$ be a set of m alternatives, $C=\left\{c_1,c_2,...,c_n\right\}$ be a set of criteria

Step 2: Construct the decision matrix $H=\left(h_{ij}\right)_{m\times n}$ where $h_{ij}\left(i=1,2,...,m\right);\left(j=1,2,...,n\right)$ is the hesitant fuzzy element of the alternatives y_i under the criteria c_j

$$H=\begin{array}{c}y_1\\y_2\\\vdots\\y_m\end{array}\begin{bmatrix}h_{11}&h_{12}&\cdots&h_{1n}\\h_{21}&h_{22}&\cdots&h_{2n}\\\vdots&\vdots&\cdots&\vdots\\h_{m1}&h_{m2}&\cdots&h_{mn}\end{bmatrix}$$

Step 3: Aggregate all the performance values $h_{ij} = (j = 1, 2, .., n)$ of the i^{th} line, and get the overall performance value h_i corresponding to the alternative y_i by HFHM operator $h_i = HFHM^{p,q}(h_{i1}, h_{i2}, ..., h_{in}) = \left(\frac{2}{n(n+1)} \left(\bigoplus_{j,k=1}^{n} (h_{ij})^p \otimes (h_{ik})^q \right) \right)^{\frac{1}{p+q}}$ where $p, q > 0$ and $i=1,2,\ldots,m$

Step 4: Calculate the score function of HFE h_i based on definition.

Step 5: Rank the alternatives based on the decreasing order of their scores. The alternative y_i with the largest score is the best alternative. If score values of two alternatives are identical then the ranking is made based on their accuracy functions.

5. NUMERICAL EXAMPLE

The purpose of investment portfolio selection is to allocate the capital to a large number of stocks to get the most profitable return. The main task of a portfolio manager is asset allocation, which is to select new assets for a new investment. Multiple criteria decision making (MCDM) method can be used to deal with the investment portfolio decision problem to improve the quality decision-making process more efficient. In general, both quantitative and qualitative data must be considered in the process of investment portfolio decision. In this paper, HFS are used to express qualitative information of each stock. In this study two criteria and three stock alternatives are only considered to avoid the computational complexity. But this can be extended to more criteria and more stock alternatives.

Suppose a fund company desires to make investment portfolio in banking sector. The company wants to evaluate fourstocks y_1, y_2, y_3, $y_4$and make an investment portfolio decision of four stocks according to two criteria namely quality ofbanking productc_1 and service quality of the banks c_2. According to the proposed method, the computational procedures of the problem are summarized as follows.

Step 1: Identify the set of available alternatives and criteria.

Let $y = \{y_1, y_2, y_3, y_4\}$ be a set of m alternatives, $C = \{c_1, c_2\}$ be a set of criteria.

Step 2: Construct the decision matrix $H = (h_{ij})_{4\times2}$ where $h_{ij}\ (i = 1, 2, 3, 4); (j = 1, 2)$ is the hesitant fuzzy element of the alternatives y_i under the criteria c_j as shown in Table 1

Table 1: Hesitant Fuzzy Matrix

	C_1	C_2
y_1	{01}	{0.2, 0.4}
y_2	{0.2, 0.3}	{0.5}
y_3	{0.4}	{0.3, 0.1}
y_4	{0.1,0.3,0.5}	{0.6,0.2}

Step 3: Aggregate all the performance values $h_{ij} = (j = 1, 2, .., n)$ of the i^{th} line, and get the overall performance value h_i corresponding to the alternative y_i by HFHM

operator $h_i = HFHM^{p,q}\left(h_{i1}, h_{i2}, \ldots, h_{in}\right) = \left(\frac{2}{n(n+1)}\left(\bigoplus_{j,k=1}^{n}\left(h_{ij}\right)^{p} \otimes \left(h_{ik}\right)^{q}\right)\right)^{\frac{1}{p+q}}$

h_1 = { 0.1530, 0.2688 }; h_2 = { 0.3667, 0.4072 }; h_3 = { 0.3517, 0.2689 } and

h_4 = {0.3981, 0.1530, 0.4671, 0.2519, 0.5518, 0.3667}

Step 4: Calculate the score function of HFE h_i , (i = 1,2,3,4) based on definition as

s(h_1) = 0.2109 ; s(h_2) = 0.3869 ; s(h_3) = 0.3103 ; s(h_4) = 0.3648

Step 5: Ranking of the alternatives based on the decreasing order of their scoresis

s(h_2)> s(h_4) >s(h_3) > s(h_1)

Hence the ranking of the alternatives is $y_2 \succ y_4 \succ y_3 \succ y_1$

The alternative y_2 with the largest score is the best alternative.

Similarly the optimal alternative based on other *GHFHM* and $GWHFH_{\omega}$ operators can be found.

6. CONCLUSIONS

The Hesitant Fuzzy Set (HFS), proposed by Torra is a powerful tool to deal with uncertainty in real situation. HM are mean type aggregation technique which is developed to reflect the interrelationship of individual criteria and HMs also focus on the aggregated arguments. In this paper, we have studied the HM under hesitant fuzzy environment, and proposed some new hesitant fuzzy aggregation operators including the HFHM, WHFHM and the GHFHM. The desirable properties, such as monotonicity, commutativity, boundedness and the special cases of the HFBM andGHFHM have been discussed in detail.

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